Quantum Interferometry using Squeezed States

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We detail a plan for an interferometer operating at the Heisenberg limit using only squeezed vacuum states. We show that we can reach the Heisenberg limit in a differential phase measurement using only squeezed vacuum inputs to an interferometer.

I. INTRODUCTION

We investigate the details of performing a differential phase measurement using only squeezed vacuum states as inputs to an interferometer.

II. THEORETICAL BACKGROUND

Here we give the theoretical background for an interferometer with squeezed vacuum input states in the ideal case. We will first consider the case of a single squeezed state passed through a phase-shifter, which is the subject of ref. [1]. We then calculate the propagation of a pair of optical states through a interferometer with a balanced homodyne readout and the expected signal from this measurement scheme. Having derived the measurement statistics from the balanced homodyne readout scheme, we will then consider the quantum Fisher information of the interferometer's output state and show that the balanced homodyne scheme is optimal in the ideal case.

A. Phase-Shift Measurements with a Single Squeezed State

This section re-derives the results of [1]. This reference calculates everything in the Fock-state basis, which is workable for the single-mode case, but still leads to a lot of ugly math. Things are much simpler if we instead use the symplectic formalism of [2, 3], and this formalism generalizes well to the multi-mode case. LIGO uses a similar formalism to [2, 3], which they call the "two photon formalism", and LIGO papers tend to emphasize frequency-dependent effects. Here, we assume everything is frequency-independent for simplicity, and postpone consideration of frequency-dependent effects until section III A.

Before investigating an interferometer with two squeezed states, we first consider a single-mode squeezed state passed through a phase-shifter and then read out by a homodyne detector, which we assume measures the state's $\hat{x}$ quadrature. Figure 1 depicts this measurement scheme.

To describe the propagation of squeezed states through an optics network, we use the symplectic formalism of refs. [2, 3]. We only consider states with zero mean, so they can be completely described by their covariance matrix $\boldsymbol{V}$. A single mode squeezed state with a squeezed amplitude quadrature has a covariance matrix given by (see appendix A 1 for details)

$$\boldsymbol{V}_{\text{sqz}} = \begin{bmatrix} \frac{e^{-r}}{2} & 0 \\ 0 & \frac{e^{r}}{2} \end{bmatrix}, \tag{2.1}$$

where r is the state's squeezing parameter. For $r = 0$, this is the covariance of a vacuum state. For $r > 0$, this is an amplitude-quadrature squeezed state and for $r < 0$, it is a phase-quadrature squeezed state.

When this squeezed state encounters a phase-shifter of ϕ radians, the phase-shifter rotates the state. The state at the phase-shifter's output, $\boldsymbol{V}_{\text{out}}$ is given by

$$\begin{aligned} \boldsymbol{V}_{\text{out}} &= \boldsymbol{R}(\phi)\boldsymbol{V}_{\text{sqz}}\boldsymbol{R}(\phi)^{\text{T}} \\ &= \begin{bmatrix} \cosh(2r) - \cos(2\phi)\sinh(2r) & \sin(2\phi)\sinh(2r) \\ \sin(2\phi)\sinh(2r) & \cosh(2r) + \cos(2\phi)\sinh(2r) \end{bmatrix}. \end{aligned} \tag{2.2}$$

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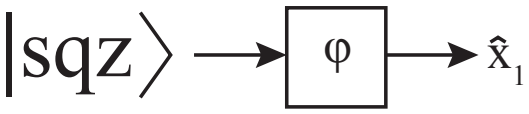


FIG. 1. A schematic showing a phase measurement with a single squeezed mode. The squeezed state is sent through a phase shifter with a phase shift ϕ and then its $\hat{x}$ quadrature is measured with a homodyne detector.

The homodyne detector measures a single quadrature of this state. In general, it will measure a quadrature $\hat{x}_\theta = \cos(\theta)\hat{x} + \sin(\theta)\hat{p}$. However, any quadrature rotation at the homodyne can be absorbed into the operating point of the phase-shifter, the mean phase-shift value that we measure small deviations around. So, without loss of generality, we assume that the homodyne measures along the $\hat{x}$ quadrature.

Since we are measuring the phase using a zero-mean squeezed state, the natural variable to use to estimate the phase is the homodyne's variance, $\langle \hat{x}^2 \rangle$. In fact, it can be shown that using the variance of a homodyne measurement is in fact the *optimal* way to estimate the phase-angle ϕ using a single-mode squeezed state [1].

For a general measurement operator, $\hat{M}$ used to estimate a parameter χ in an experiment, the measurement's signal to noise ratio (SNR), taken to be the inverse of the precision in measuring ϕ, is given by [4, 5]

$$\text{SNR} = \frac{1}{\langle \phi^2 \rangle} \equiv \frac{\left|\partial_\chi \langle \hat{M} \rangle\right|}{\sqrt{\langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2}}. \tag{2.3}$$

For our experiment, we take the estimator to be $\hat{M} = \hat{x}^2$ so that the SNR is given by

$$\text{SNR} = \frac{\left|\partial_\phi \langle \hat{x}^2 \rangle\right|}{\sqrt{\langle \hat{x}^4 \rangle - \langle \hat{x}^2 \rangle^2}} = \frac{\left|\partial_\phi \text{Var}[\hat{x}]\right|}{\sqrt{2}\text{Var}[\hat{x}]}, \tag{2.4}$$

where we have used Isserlis' theorem and $\text{Var}[\hat{x}] = \hat{x}^2$ to get the final expression. Isserlis' theorem states that for a zero-mean Gaussian random variable, $\hat{A}$, $\langle \hat{A}^4 \rangle = \langle \hat{A}^2 \rangle^2$.

Using eq. (2.2), we can explicitly compute the system's SNR for estimating the pahse shift ϕ. We find

$$\text{SNR} = \frac{\sqrt{2}\sinh(2r)|\sin(2\phi)|}{\cosh(2r) - \sinh(2r)\cos(2\phi)}, \tag{2.5}$$

which is maximized for

$$\phi = \frac{1}{2}\cos^{-1}[\tanh(2r)]. \tag{2.6}$$

With this value of ϕ, the SNR is

$$\text{SNR} = \sqrt{2}\sinh(2r). \tag{2.7}$$

The number of photons in a squeezed state, $\bar{n}$, with squeezing parameter r is given by eq. (A3); $r = \sinh^{-1}\sqrt{\bar{n}}$. Plugging this into eq. (2.7), we find that the SNR in terms of the mean photon number is given by

$$\text{SNR} = \sqrt{8}\sqrt{\bar{n}^2 + \bar{n}}, \tag{2.8}$$

which has Heisenberg-scaling, i.e. the SNR increases linearly with the photon number $\bar{n}$, for large $\bar{n}$. This contrasts a typical interferometer, which only achieves shot noise scaling: $\text{SNR} \propto \sqrt{\bar{n}}$ for large $\bar{n}$. We have now seen that a single squeezed vacuum state can measure a phase with Heisenberg-limited uncertainty. In the next section, we will consider using a pair of squeezed states in an interferometer to reach the Heisenberg limit for measuring common-mode and differential phase shifts.

B. An Ideal Interferometer with Squeezed Vacuum Inputs

Having considered using a single squeezed state to make a phase measurement, we now consider using a pair of squeezed states to make a differential phase measurement. Figure 2 shows the experimental scheme. The interferometer's input is a two-mode squeezed state with the second squeezed state squeezed at a phase-angle θ_s relative to the

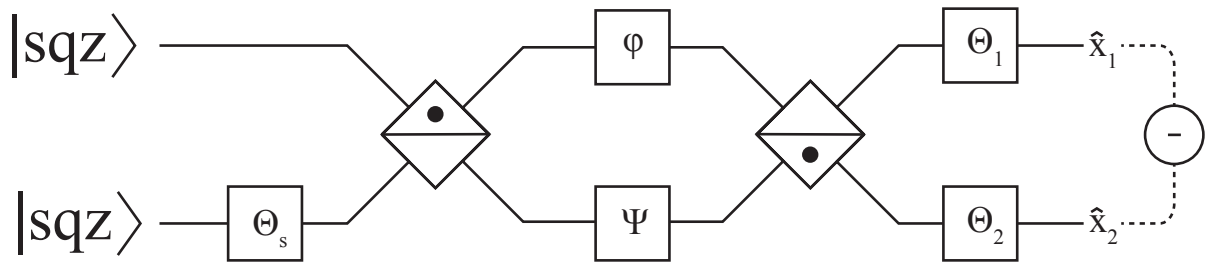


FIG. 2. A schematic showing an interferometer with a pair of squeezed states as inputs. A pair of squeezed states with a relative phase-angle θ_s are launched into the interferometer. The arms of the interferometer receive phase-shifts ϕ and ψ. A pair of homodyne detectors then measure the interferometer's output states at angles θ_1 and θ_2. This last step is equivalent to first rotating the output states by phase-angles θ_1 and θ_2 and then measuring the position quadrature of each mode. The homodyne outputs are then subtracted and squared to derive information about the interferometer's differential phase, $\phi_d = \phi - \psi$.

first. The covariance matrix at the interferometer's input is then

$$\boldsymbol{V}_{\rm in} = \begin{bmatrix} \boldsymbol{V}_{\rm sqz} & 0 \\ 0 & \boldsymbol{R}(\theta_s)\boldsymbol{V}_{\rm sqz}\boldsymbol{R}(\theta_s)^{\rm T} \end{bmatrix} \tag{2.9}$$

where the superscript T denotes a matrix transpose and the single-mode rotation matrix $\boldsymbol{R}(\theta)$ is defined by

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{2.10}$$

The input state described by eq. (2.9) is sent into the interferometer. It first hits a balanced beam splitter, which has an associated symplectic matrix given by

$$\boldsymbol{S}_{\rm bs} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{2.11}$$

After the first beam splitter, the state is given by

$$\boldsymbol{V}_{\rm bs} = \boldsymbol{S}_{\rm bs}\boldsymbol{V}_{\rm in}\boldsymbol{S}_{\rm bs}^{\rm T} \tag{2.12}$$

Each mode inside the interferometer then accrues a phase shift. Denoting the phase shift on the first mode by ϕ and the phase shift on the second mode as ψ, the state evolves to

$$\boldsymbol{V}_{\rm ph} = \begin{bmatrix} \boldsymbol{R}(\phi) & 0 \\ 0 & \boldsymbol{R}(\psi) \end{bmatrix} \boldsymbol{V}_{\rm bs} \begin{bmatrix} \boldsymbol{R}(\phi)^{\rm T} & 0 \\ 0 & \boldsymbol{R}(\psi)^{\rm T} \end{bmatrix}. \tag{2.13}$$

After the phase shifts, the modes are recombined on a balanced beamsplitter and the output state has a covariance matrix given by

$$\boldsymbol{V}_{\rm out} = \boldsymbol{S}_{\rm bs}^{\rm T}\boldsymbol{V}_{\rm ph}\boldsymbol{S}_{\rm bs}. \tag{2.14}$$

We then read out this output state with a pair of homodyne detectors, each measuring an arbitrary quadrature of one of the two outgoing modes. This arbitrary quadrature readout is equivalent to first rotating each of the modes and then reading out the position quadrature on each homodyne detector. Applying these rotations, the homodynes measure the position quadratures of the state with covariance matrix

$$\boldsymbol{V}^{\rm meas} = \begin{bmatrix} \boldsymbol{R}(\theta_1) & 0 \\ 0 & \boldsymbol{R}(\theta_2) \end{bmatrix} \boldsymbol{V}_{\rm out} \begin{bmatrix} \boldsymbol{R}(\theta_1)^{\rm T} & 0 \\ 0 & \boldsymbol{R}(\theta_2)^{\rm T} \end{bmatrix}. \tag{2.15}$$

The first homodyne reads out a quadrature at angle θ_1 to the first mode's position quadrature axis, $\hat{x}_1$, which has zero mean and a variance of

$$\begin{aligned} \boldsymbol{V}_{1,1}^{\rm meas} = &\sin^2\left(\frac{\phi_d}{2}\right)\left[\cosh(2r) + \cos(2\theta_1 + 2\theta_s + \phi_c)\sinh(2r)\right] \\ &+ \cos^2\left(\frac{\phi_d}{2}\right)\left[\cosh(2r) - \cos(2\theta_1 + \phi_c)\sinh(2r)\right], \end{aligned} \tag{2.16}$$

where we have introduced the differential phase $\phi_\mathrm{d} \equiv \phi - \psi$ and the common-mode phase $\phi_\mathrm{c} \equiv \phi + \psi$. The second homodyne reads out a quadrature at angle θ_2 to the first mode's position quadrature axis, $\hat{x}_2$, which has zero mean and a variance of

$$\begin{aligned}\boldsymbol{V}_{3,3}^{\mathrm{meas}} =& \sin^2\left(\frac{\phi_\mathrm{d}}{2}\right)\left[\cosh(2r) + \cos(2\theta_2 + \phi_\mathrm{c})\sinh(2r)\right] \\ &+ \cos^2\left(\frac{\phi_\mathrm{d}}{2}\right)\left[\cosh(2r) - \cos(2\theta_2 + 2\theta_\mathrm{s} + \phi_\mathrm{c})\sinh(2r)\right]\end{aligned} \tag{2.17}$$

The covariance between these two measurements is given by

$$\boldsymbol{V}_{1,3}^{\mathrm{meas}} = \boldsymbol{V}_{3,1}^{\mathrm{meas}} = \sin(\phi_\mathrm{d})\cos(\theta_\mathrm{s})\sin(\theta_1 + \theta_2 + \theta_\mathrm{s} + \phi_\mathrm{c})\sinh(2r). \tag{2.18}$$

We now consider measuring the difference of square the two homodyne detectors' outputs, i.e. measuring $\hat{M}_\mathrm{sqz} = \hat{x}_1^2 - \hat{x}_2^2$. The resulting signal is given by

$$\begin{aligned}\langle \hat{M}_\mathrm{sqz}\rangle &= V_{1,1}^{\mathrm{meas}} - V_{3,3}^{\mathrm{meas}} \\ &= 2\sinh(2r)\sin(\theta_\mathrm{c} + \theta_\mathrm{s} + \phi_\mathrm{c})\big[\cos(\theta_\mathrm{s})\cos(\phi_\mathrm{d})\sin(\theta_\mathrm{d}) - \cos(\theta_\mathrm{d})\sin(\theta_\mathrm{s})\big]\end{aligned} \tag{2.19}$$

To compute $\langle \hat{M}_\mathrm{sqz}\rangle$, we will want to use Isserlis' theorem several times. We will want to use $\langle x_i^4\rangle = 3\langle x_i^2\rangle^2$ and $\langle x_i^2 x_j^2\rangle = \langle x_i^2\rangle\langle x_j^2\rangle + 2\langle x_i x_j\rangle^2$ for zero-mean Gaussian random variables x_i and x_j. We will also make use of the fact that the two homodyne operators $\hat{x}_1$ and $\hat{x}_2$ commute. With this in mind, we find

$$\langle \hat{M}_\mathrm{sqz}^2\rangle = \left(V_{1,1}^{\mathrm{meas}}\right)^2 + \left(V_{3,3}^{\mathrm{meas}}\right)^2 - 2\left(V_{1,3}^{\mathrm{meas}}\right)^2 \tag{2.20}$$

The analytic expression for $\langle \hat{M}_\mathrm{sqz}^2\rangle$ is rather complicated without further simplifying assumptions, so we omit it here.

1. *Identical Squeezed States*

We will now specialize to consider the case of injecting identical squeezed states into the interferometer. Assuming that we inject identical squeezed states is equivalent to setting $\theta_\mathrm{s} = 0$.

We will see that injecting identical squeezed states gives us maximal differential phase sensitivity at the expense of having no entanglement inside the interferometer. To achieve optimal differential phase sensitivity, we choose the following parameter values:

$$\begin{aligned}\theta_\mathrm{s} &= 0 \\ \theta_\mathrm{d} &= \pi/2 \\ \phi_\mathrm{d} &= \pi/2 \\ \phi_\mathrm{c} + \theta_\mathrm{c} &= \pi/2\end{aligned} \tag{2.21}$$

As can be seen from the last equation here, only the sum $\phi_\mathrm{c} + \theta_\mathrm{c}$ contributes to the SNR for the identical squeezed state case.

With these parameter values, the SNR is given by

$$\mathrm{SNR} = \frac{|\partial_{\phi_\mathrm{d}}\langle \hat{M}_\mathrm{sqz}\rangle|}{\sqrt{\langle \hat{M}_\mathrm{sqz}^2\rangle}} = \sinh(2r) = \sqrt{\bar{n}^2 + 2\bar{n}}, \tag{2.22}$$

where $\bar{n} = 2\sinh^2(r)$ is the total number of photons incident on the interferometer. For large $\bar{n}$, this SNR grows linearly with $\bar{n}$ and achieves Heisenberg scaling.

Additionally, we dinf that for these parameter values, $\partial_{\phi_\mathrm{c}}\langle \hat{M}_\mathrm{sqz}\rangle = 0$ and the interferometer is first-order insensitive to common-mode phases.

We now trace the state's evolution through the interferometer. We use the parameter values of eq. (2.21) with the exception of $\phi_\mathrm{d} = \pi/2 + \delta\phi$, which captures the small differential phase being measured, and the assumption that $\phi_\mathrm{c} = 0$ and $\theta_\mathrm{c} = \pi/2$, which lifts the degeneracy between these parameters.

The state sent into the interferometer is

$$\boldsymbol{V} = \frac{1}{2}\begin{bmatrix} e^{-2r} & 0 & 0 & 0 \\ 0 & e^{2r} & 0 & 0 \\ 0 & 0 & e^{-2r} & 0 \\ 0 & 0 & 0 & e^{2r} \end{bmatrix}. \tag{2.23}$$

After the first beam splitter, the state is unchanged. After the differential phase-shift, the state becomes

2. EPR Entangled States

If instead of using a pair of identical squeezed states as the interferometer's input states we use a pair of squeezed states squeezed along oposite quadrature axes, we create an EPR state after the first beam splitter. As we will see, using an EPR state causes us to lose any differential phase sensitivity, but we gain a maximally path-entangled state inside the interferometer. This state is potentially useful for investigating the inteaction between highly entangled optical states and gravity.

The covariance matrix of the state sent into the interferometer is

$$\boldsymbol{V} = \frac{1}{2}\begin{bmatrix} e^{-2r} & 0 & 0 & 0 \\ 0 & e^{2r} & 0 & 0 \\ 0 & 0 & e^{2r} & 0 \\ 0 & 0 & 0 & e^{-2r} \end{bmatrix}. \tag{2.24}$$

After the first beam splitter, it becomes

$$\boldsymbol{V} = \begin{bmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{bmatrix}, \tag{2.25}$$

which is a maximally entangled EPR state.

The state then encounters a phase shift on each interferometer arm, after which it becomes

$$\boldsymbol{V} = \begin{bmatrix} \cosh(2r) & 0 & \sinh(2r)\cos(\phi_c) & -\sinh(2r)\sin(\phi_c) \\ 0 & \cosh(2r) & -\sinh(2r)\sin(\phi_c) & -\sinh(2r)\cos(\phi_c) \\ \sinh(2r)\cos(\phi_c) & -\sinh(2r)\sin(\phi_c) & \cosh(2r) & 0 \\ -\sinh(2r)\sin(\phi_c) & -\sinh(2r)\cos(\phi_c) & 0 & \cosh(2r) \end{bmatrix}. \tag{2.26}$$

The state then encounters the interferometer's output beam splitter, after which its covariance matrix is given by

$$\boldsymbol{V} = \begin{bmatrix} \cosh(2r) - \sinh(2r)\cos(\phi_c) & \sinh(2r)\sin(\phi_c) & 0 & 0 \\ \sinh(2r)\sin(\phi_c) & \cosh(2r) + \sinh(2r)\cos(\phi_c) & 0 & 0 \\ 0 & 0 & \cosh(2r) + \sinh(2r)\cos(\phi_c) & -\sinh(2r)\sin(\phi_c) \\ 0 & 0 & -\sinh(2r)\sin(\phi_c) & \cosh(2r) - \sinh(2r)\cos(\phi_c) \end{bmatrix}. \tag{2.27}$$

We see that at the interferometer's output, the state is no longer entangled. Comparing eq. (2.27) to eq. (2.2), we see that the output state is a pair of unentangled squeezed states with oposite rotations from the interferometer's common-mode phase. Notably, the state is completely independent of the interferometer's differential phase, in contrast to interferometry with NOON states, which is only sensitive to differential phases.

C. Comparison to Other Input States

Having derived the SNR for a pair of squeezed vacuum modes in an interferometer, we now compare the SNR for that scheme to the SNR of several other standard schemes.

1. Squeezed Vacuum on a Single Input Port

Having considered the case where we inject squeezed vacuum onto both interferometer input ports, we consider injecting squeezed vacuum onto a single input port, with the other kept as a vacuum mode. The input covariance matrix is then given by

$$\boldsymbol{V}_{\text{in}} = \frac{1}{2}\begin{bmatrix} e^{-r} & 0 & 0 & 0 \\ 0 & e^{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.28}$$

2. Coherent State and Squeezed Vacuum

Here, we consider injecting squeezed vacuum on one port of an interferometer and a coherent state on the other. We treat this problem with the symplectic formalism of [2, 3] as this considerably simplifies the math compared to working in the Heisenberg picture with lowering operators.

Unlike the case with

3. NOON States

4. Thermal States

III. PRACTICALITIES

A. Effects of a Finite Squeezing Bandwidth

For an analysis of spectrum analyzer operation, see Appendix A.3 of [6].

B. Effects of Optical Losses

C. Effects of Asymmetric Squeezing

D. Effects of Phase Noise

IV. MEASUREMENT SCHEME FOR FINITE BANDWIDTH SQUEEZED STATES

V. OPTICAL LAYOUT

A. Alignment

To maximize the visibility through the interferometer, we'll want to exactly match the optical path lengths on the two arms of the interferometer. To do this, we'll want to achieve rough alignment with a ruler and a laser beam and then achieve exact path length matching using a white light source as in the Michelson-Morley experiment [7]. When the path lengths of the arms are exactly matched, we should see white-light interference fringes.

VI. INTERFEROMETER CONTROLS

A. Sensing Noise and Minimal Achievable Phase Noise

We want to use the CLF fields to lock the interferometer's differential phase. This could be done using the CLF voltage signals out of the homodyne detectors, but these voltage levels are inherently sensitive to the LO power levels and the relative phase between the CLF and LO beams. We can eliminate this phase sensitivity by squaring the

voltage signal, but this cannot be done easily with a Red Pitaya or with analog electronics. Instead, we will use a 1% pickoff of each beam at the output of the interferometer and send these signals to a differential receiver with a large transimpedance gain. The problem with this approach is that the CLF beams are weak, they are each about 1 μW, so the pickoffs will be about 10 nW. Here, we show that this scheme should allow us to suppress interferometer phase fluctuations down to the 1 mrad RMS level.

First, we estimate the electronics noise of the homodyne circuit. We assume we will use a JFET operational amplifier such as the TI OPA828, which has an input current noise of about 1 fA/$\sqrt{\text{Hz}}$, and an input voltage noise of about 4 nV/$\sqrt{\text{Hz}}$. Assuming we use a 10 MΩ transimpedance resistor, this current noise leads to an equivalent voltage noise of about 10 nV/$\sqrt{\text{Hz}}$. The thermal noise of this resistor will be (note that all spectral densities are single-sided)

$$\sqrt{S_{VV}^{\text{th}}} = \sqrt{4k_{\text{B}}TR} = 400 \text{ nV}/\sqrt{\text{Hz}}, \tag{6.1}$$

which will dominate over the operational amplifier's noise.

The photocurrent, i from a photodetector with quantum efficiency η and photon flux $\dot{n}$ is

$$i = \eta e \dot{n}, \tag{6.2}$$

where e is the elementary electric charge. Assuming the light on the photodetector is nearly monochromatic with frequency ω_l, the photon flux is related to this frequency, Planck's constant, $\hbar$, and the optical power, P, by $\dot{n} = P/(\hbar\omega_l)$. Defining the photodetector's responsivity as $R \equiv /(\hbar\omega_l)$, and assuming that the photocurrent is fed through a transimpedance amplifier with gain (or resistance) g the resulting voltage is given by

$$V = g\eta RP, \tag{6.3}$$

so the output voltage spectral density from shot noise is related to the incident power spectral density as

$$S_{VV}^{\text{shot}} = (g\eta R)^2 S_{PP}. \tag{6.4}$$

To determine the value of S_{PP} for a given optical power, we use the fact that the photon flux in a laser beam is $P/(\hbar\omega_l)$ and q^2, where q is the amplitude quadrature and we have assumed $q \gg 1$ so we can ignore the constant term from vacuum fluctuations. Denoting the mean value of a fluctuating parameter x as $\bar{x}$ and its fluctuations as δx, we then have

$$\bar{P} = \hbar\omega_l \bar{q}^2 \tag{6.5}$$

and

$$\delta P = 2\hbar\omega_l \bar{q}\delta q = 2\sqrt{\hbar\omega_l \bar{P}}\delta q, \tag{6.6}$$

where we have linearized and neglected the small terms quadratic in the fluctuations. Using $S_{qq} = 1$, we can then write shot noise as

$$S_{VV}^{\text{shot}} = 2(g\eta R)^2 \hbar\omega_l P. \tag{6.7}$$

Assuming a beam power of 10 nW, a transimpedance gain of 10 MΩ, a 1064 nm beam, and unit quantum efficiency, we have

$$\sqrt{S_{VV}^{\text{shot}}} = 525 \text{ nV}/\sqrt{\text{Hz}} \tag{6.8}$$

per diode. Combining the noise from both diodes, we have

$$\sqrt{S_{VV}^{\text{shot}}} = 740 \text{ nV}/\sqrt{\text{Hz}}. \tag{6.9}$$

Comparing this noise level to the electronics noise given by eq. (6.1), we see that the detector is shot noise limited, although shot noise is the same order of magnitude as the electronics noise.

Assuming we have a shot noise limited detector, as we have just seen is possible, we should be able to suppress phase fluctuations down to the level dictated by shot noise. We call the photodiode that produces an increasingly positive output voltage with increasing power the "positive" diode, and the photodiode that produces a decreasing

output voltage with increasing power the "negative" diode, and label these powers by P_+ and P_- respectively. These powers are related to the input power, P by

$$\begin{aligned} P_+ &= P\cos^2(\phi_\mathrm{d}/2) \\ P_- &= P\sin^2(\phi_\mathrm{d}/2), \end{aligned} \tag{6.10}$$

where ϕ_d is the interferometer's differential phase as defined in section II B. The differential receiver's output voltage is then

$$V = g\eta R\,(P_+ - P_-) = g\eta RP\cos(\phi_\mathrm{d}). \tag{6.11}$$

Linearizing about the lock point of $\phi_\mathrm{d} = \pi/2$, we have

$$\delta V = g\eta RP\delta\phi_\mathrm{d}, \tag{6.12}$$

so the differential receiver's output voltage fluctuations are related to interferometer phase fluctuations by

$$S_{VV} = (g\eta RP)^2 S_{\phi_\mathrm{d}\phi_\mathrm{d}}. \tag{6.13}$$

Assuming these fluctuations are suppressed to the shot noise level by appropriate feedback, we have

$$S^\mathrm{min}_{\phi_\mathrm{d}\phi_\mathrm{d}} = \frac{S^\mathrm{shot}_{VV}}{(g\eta RP)^2} = \frac{4\hbar\omega_l}{P}. \tag{6.14}$$

where the factor of 4 accounts for the shot noise on each diode. For a 10 nW, 1064 nm beam, the resulting phase fluctuation level is

$$S^\mathrm{min}_{\phi_\mathrm{d}\phi_\mathrm{d}} = 7\times10^{-11}\ \mathrm{rad}^2/\mathrm{Hz}. \tag{6.15}$$

Assuming a noise bandwidth of 10 kHz, the root-mean-squared phase noise level is then

$$\sqrt{\langle\delta\phi_\mathrm{d}^2\rangle} = \sqrt{S^\mathrm{min}_{\phi_\mathrm{d}\phi_\mathrm{d}}\Delta f} = 0.8\ \mathrm{mrad}. \tag{6.16}$$

With the expected optical signal levels and electronics noise and an optimal filter, we should be able to suppress the interferometer's phase noise down to about 1 mrad. In practice, our feedback filter will not be optimal, and our phase noise level will exceed this value. Since we want to limit phase noise to the 10 mrad level, we will need to be careful to design a nearly optimal filter for this feedback loop.

B. Error Signal and Lock Point

As discussed in section II B, we will want to lock the interferometer such that its differential phase is $\phi_\mathrm{d} = \pi/2$. For coherent laser beam with power P injected into a single input of the interferometer, the output power levels at the two output ports will be

$$\begin{aligned} P_+ &= P\cos^2(\phi_\mathrm{d}/2) \\ P_- &= P\sin^2(\phi_\mathrm{d}/2), \end{aligned} \tag{6.17}$$

and the differential receiver's output voltage will be

$$V = g\eta R\,(P_+ - P_-) = g\eta RP\cos(\phi_\mathrm{d}). \tag{6.18}$$

If we lock to $\phi_\mathrm{d} = \pi/2$, we then have equal powers in the interferometer's output ports at the lock point, and the error signal will have a mean value of zero and a slope of

$$\frac{dV}{d\phi_d} = -g\eta RP. \tag{6.19}$$

This scheme also has the benefit that it is first-order insensitive to power fluctuations on the CLF beams used for the lock.

C. Alternative Locking Schemes

Instead of using 1% pickoffs of the interferometer's output ports, we propose two alternative locking schemes. The first is to use the magnitude of the CLF beatnote on the homodyne detectors to lock the interferometer. The upside of this scheme is that it can be done with existing optics, but the downside is that the lock will be sensitive to LO power fluctuations on both homodynes and that it requires more complicated controls than we can currently implement with a Red Pitaya. The control scheme can be implemented with a Moku Pro. The second is to lock the interferometer in green. This has the advantage of allowing us to lock to a substantially brighter beam and filter out the beam used for the interferometer lock with dichroics instead of 1% pickoffs. The downsides are that this scheme will require us to order new superpolished optics and it will be exposed to any differences between 532 nm and 1064 nm light in the interferometer.

1. Read CLF Tones from Output Homodynes

In the scheme of using the CLF tones read off of the homodynes, we would read out the CLF tone from each homodyne, square the tone to remove information about the relative phase between the CLF and LO beam, and then subtract the squared signals to generate an error signal. This differential signal will be proportional to the power imbalance of the two interferometer output ports. It would then be sent to a PID block and used to servo the interferometer's phase. The downside of this strategy is twofold. First, the magnitude of the CLF tone depends on both the CLF power and the LO power levels on each homodyne. Using this scheme then requires high levels of passive LO stability, or an LO with actively stabilized LO power. The second downside is that this cannot be done with a Red Pitaya and pyrpl. It can be done with a Moku Pro out of the box by using lock-in amplifier blocks to square the CLF voltages and a PID block to subtract these signals and generate the feedback signal from the error signal. However, a Moku Pro costs about $20,000 and Moku company representatives say that this cannot be done at our CLF frequencies with a Moku Go or Moku Lab.

2. Lock with a Green Beam

In the 532 nm locking scheme, we would inject a 532 nm beam through a dichroic beam splitter and use this to lock the interferometer. Like the IR locking scheme with 1% pickoffs, this scheme is insensitive to 532 nm power fluctuations, however it is exposed to any relative fluctuations in the interferometer's behavior at 532 nm vs 1064 nm, for instance from pressure fluctuations changing the relative indices of refraction at these wavelengths. Additionally, it will require new optics. At a minimum, this scheme will require us to purchase a pair of 50/50 beam splitters coated for both 532 nm and 1064 nm and a pair of dual wavelength half-inch mirrors for the phase-shifter. Accounting for backups, and assuming that each optic costs about $1,000, which is in line with previous quotes from Intrinsic Crystal, this scheme would cost about $8,000.

VII. SQUEEZE ANGLE CONTROL

A. Relative Phase Angle Measurement Schemes

We consider two schemes for controlling the relative squeezing angle between the two OPO's outputs on the first beam splitter. In the first, shown in fig. 3(a), a 1% pickoff of the interferometer light is directly measured by a radio frequency photodetector (RFPD). In the second, shown in fig. 3(b), the light picked off from the interferometer is first interfered with a strong local oscillator beam and then detected on a RFPD. This second scheme is more complicated than the first, but allows for much better noise performance, as we will see.

1. Squeeze Angle Control with Direct Detection

Here, we consider a scheme that directly monitors the beat note between the two CLF tones on a RFPD as shown in fig. 3(a). We assume that the power in the i$^{\text{th}}$ CLF beam incident on the interferometer is P_i. If the power on the RFPD is then

$$P_{\text{RFPD}} = P_1' \cos(\omega_1 t + \phi_1)^2 + P_2' \cos(\omega_2 t + \phi_2)^2 + 2\sqrt{P_1' P_2'} \cos(\omega_1 t + \phi_1) \cos(\omega_2 t + \phi_2), \tag{7.1}$$

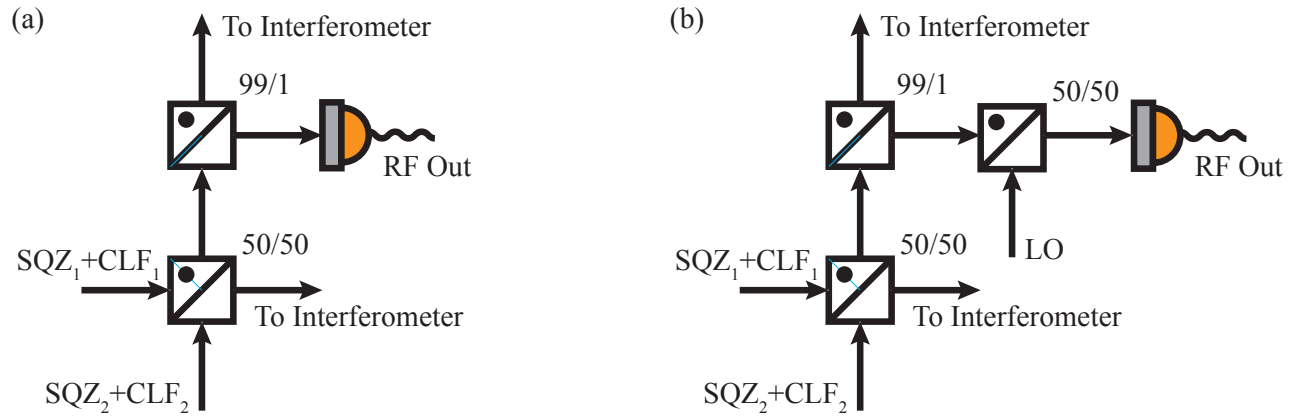


FIG. 3. Methods of monitoring the relative squeezing angle between the two fields on the first beam splitter. (a) the beat note between the two CLF beams is monitored directly on a radio frequency photodetector (RFPD). (b) the CLF beams are first amplified using a local oscillator with a power of several mW before impinging the RFPD.

where P_i' is the amount of power from the i$^{\text{th}}$ CLF impinging on the PD (a few nW). We assume that the voltage signal is related to the power incident on the RFPD as $V_{\text{out}} = GP_{\text{RFPD}}$. After band-passing the resulting signal with a pass band centered on the frequency $|\omega_1 - \omega_2|$, the output voltage of the RFPD is

$$V_{\text{out}} = G\sqrt{P_1' P_2'}\cos\big((\omega_1 - \omega_2)t + (\phi_1 - \phi_2)\big). \tag{7.2}$$

To derive an error signal, we will then demodulate this output voltage at the frequency $\omega_1 - \omega_2$ and apply a low pass filter. After multiplying by a tone $V_{\text{LO}} = \cos\big((\omega_1 - \omega_2)t + \phi_{\text{demod}}\big)$, the error signal is

$$V_{\text{err}} = \frac{G}{2}\sqrt{P_1' P_2'}\cos(\phi_1 - \phi_2 - \phi_{\text{demod}}). \tag{7.3}$$

Choosing to servo this error signal to zero, we see that voltage and phase fluctuations are related as

$$\delta V_{\text{err}} = \frac{G}{2}\sqrt{P_1' P_2'}\,\delta\phi, \tag{7.4}$$

such that

$$S_{VV}^{\text{err}} = \left(\frac{G}{2}\right)^2 P_1' P_2' S_{\phi\phi}, \tag{7.5}$$

which we can use to convert between voltage noise and phase noise.

We assume that we use a Newfocus 1811 RFPD, which has a responsivity of $\eta \times R \approx 0.7$ A/W, a transimpedance gain of $g = 4 \times 10^4$ V/A such that $G = \eta \times R \times g = 2.8 \times 10^4$ V/W and an output voltage noise of -120 dBm/Hz $= (225\text{ nV})^2$/Hz. Assuming that the total CLF power is about 10 nW, the show noise of this RFPD will be $S_{VV}^{\text{shot}} = G^2\hbar\omega P = 1.5\text{ nV}^2$/Hz, which is much less than the detector's output voltage noise level. As a result, this detection scheme will be limited by electronics noise at a level of about $(225\text{ nV})^2$/Hz. Using eq. (7.6), we can convert this to an equivalent phase noise and find

$$S_{\phi\phi}^{\text{min}} = 6 \times 10^{-9}\,\frac{\text{rad}^2}{\text{Hz}} \tag{7.6}$$

Assuming a noise bandwidth of $\Delta f = 10$ kHz, we will then have a minimal RMS phase noise level of

$$\delta\phi_{\text{RMS}} = \sqrt{S_{\phi\phi}^{\text{min}}\Delta f} = 7\text{ mrad}. \tag{7.7}$$

This RMS phase noise level is optimistic as it assumes that our feedback loop is perfectly optimized to suppress phase fluctuations down to the shot noise level. We will see in the next section that we can do much better by making the scheme slightly more complicated and amplifying the CLF beams with a strong local oscillator beam.

2. Squeeze Angle Control with LO Amplification

Instead of directly detecting the CLF signal on the RFPD, we can first amplify the CLF beam with a strong local oscillator beam with several mW of optical power as depicted in fig. 3(b). In this case, the power on the RFPD is given by

$$P_{\mathrm{RFPD}} = \left(\sqrt{P_1'}\cos(\omega_1 t + \phi_1) + \sqrt{P_2'}\cos(\omega_2 t + \phi_2) + \sqrt{P_{\mathrm{LO}}'}\cos(\omega_{\mathrm{LO}} t + \phi_{\mathrm{LO}})\right)^2, \tag{7.8}$$

where $P_i' = R * P_i/4$ is the amount of CLF power that makes it to the RFPD and $P_{\mathrm{LO}}' = P_{\mathrm{LO}}/2$ is the amount of LO power that makes it to the RFPD. This equation contains six terms, but since $P_{\mathrm{LO}}' \gg P_i'$, we can neglect all terms that do not contain P_{LO}', so

$$P_{\mathrm{RFPD}} \approx P_{\mathrm{LO}}'^2\cos(\omega_{\mathrm{LO}} t + \phi_{\mathrm{LO}})^2 + \sqrt{P_{\mathrm{LO}}' P_1'}\cos(\omega_{\mathrm{LO}} t + \phi_{\mathrm{LO}})\cos(\omega_1 t + \phi_1) + \sqrt{P_{\mathrm{LO}}' P_2'}\cos(\omega_{\mathrm{LO}} t + \phi_{\mathrm{LO}})\cos(\omega_2 t + \phi_2). \tag{7.9}$$

The first term here will determine the shot noise level, which is given by

$$S_{VV}^{\mathrm{shot}} = G^2 \hbar\omega P_{\mathrm{LO}}'. \tag{7.10}$$

Assuming we use a Newfocus 1811 with $g = 4\times10^4$ V/A and $\eta \times R = 0.7$ A/W and have a LO power of 1 mW, the shot noise level is $S_{VV}^{\mathrm{shot}} = 0.145\ \mu\mathrm{V}^2/\mathrm{Hz}$, which is much larger than the detector's electronics noise of 225 $\mathrm{nV}^2/\mathrm{Hz}$ and will be the dominant noise source.

With the LO beam, we will no longer demodulate at the difference frequency of the CLF tones, but at the difference frequencies between the LO and CLF tones. As in the previous section, this will lead to an error signal voltage related to phase fluctuations as

$$\delta V_{\mathrm{err}} = \frac{G}{2}\sqrt{P_i' P_{\mathrm{LO}}'}\,\delta\phi \tag{7.11}$$

such that

$$S_{VV}^{\mathrm{err}} = \frac{G^2}{4} P_i' P_{\mathrm{LO}}' S_{\phi\phi}. \tag{7.12}$$

For a shot noise limited detector, S_{VV}^{err} will be determined by shot noise such that the minimal detectable phase fluctuations are

$$S_{\phi\phi}^{\mathrm{min}} = \frac{4\hbar\omega}{P_i'}. \tag{7.13}$$

Assuming a CLF power incident on the RPFD of $\Delta f = 10$ nW and a feedback bandwidth of 10 kHz, we have

$$\delta\phi_{\mathrm{RMS}} = \sqrt{\frac{4\hbar\omega\Delta f}{P_i'}} = 0.6\ \mathrm{mrad}. \tag{7.14}$$

This is better than the scheme above, but not by much. It also assumes perfect visibility, if we instead have 75% visibility (equivalent to about 50% loss), we will have $P_i' \to P_i'/2$ and $\delta\phi_{\mathrm{RMS}} = 0.9$ mrad.

To continue to improve the SNR beyond this level, we need to decrease the transimpedance gain while remaining shot noise limited, or increase the CLF power. We could also switch to using a differential receiver, which would suppress shot noise from the LO beam and achieve much better SNR in principle, at the expense of further complicating the setup.

B. Demodulation Angle Optimization

To appropriately control the relative squeezing angles, it is essential to be able to quickly determine the relative angle between the two squeezed states and set this to the desired value by adjusting the demodulation phase. Here, we explore how this can be done.

Once the interferometer is locked, the differential phase will be servoed to $\phi_d \to \pi/2$ such that the variance measured on each homodyne will be given by (from inserting $\phi_d = \pi/2$ into eqs. (2.16) and (2.17))

$$\boldsymbol{V}_{i,i}^{\text{meas}} = \cosh(2r) + \sinh(2r)\cos(\theta_{\text{s}})\cos(2\theta_i + \phi_{\text{c}} + \theta_{\text{s}}) \tag{7.15}$$

where $i \in \{1, 3\}$. If we then scan the homodyne LO phase to scan the readout angle, θ_i, at a frequency ω_ℓ, the variance will change as

$$\boldsymbol{V}_{i,i}^{\text{meas}}(t) = \cosh(2r) + \sinh(2r)\cos(\theta_{\text{s}})\cos(\omega_\ell t + \phi). \tag{7.16}$$

The amplitude of $\boldsymbol{V}_{i,i}^{\text{meas}}(t)$ is then maximized when $\theta_{\text{s}} = n\pi$ where $n \in \mathbb{Z}$ is an integer. By adjusting the demodulation phase on the squeeze angle control loop until the amplitude of $\boldsymbol{V}_{i,i}^{\text{meas}}(t)$ is maximized, we can then be sure that we are injecting squeezed states with the same squeezing axis into the interferometer. Note that squeezed states are invariant under rotations of π radians, so we need not worry about whether we have $\theta_{\text{s}} = 0$ or $\theta_{\text{s}} = \pi$.

The upshot of this technique for optimizing the relative squeezing angle is that $\boldsymbol{V}_{i,i}^{\text{meas}}(t)$ can be read off of a spectrum analyzer in real time. We can then optimize θ_{s} by looking at a single curve in real time with no additional analysis or calibration.

VIII. ADJUSTING THE LOCAL OSCILLATOR DEMODULATION PHASES

Before locking the LO loops, the interferometer and relative squeeze angle loops should be closed so we have $\phi_{\text{d}} = \pi/2 + \delta\phi_{\text{d}}$ and $\phi_{\text{s}} = \delta\phi_{\text{s}}$ where $\delta\phi_{\text{d}}, \delta\phi_{\text{s}} \ll 1$. When locking the LO loops, we then want $\theta_{\text{d}} = \pi/2$ and $\theta_{\text{c}} = \pi/2 - \phi_{\text{c}}$ or equivalently, $\theta_1 = \pi/2 - \phi_{\text{c}}/2$ and $\theta_2 = -\phi_{\text{c}}/2$. Plugging these values into eqs. (2.16) and (2.17), we see that this corresponds to maximum anti-squeezing on the first homodyne and maximum squeezing on the second. The appropriate demodulation phases can then be found by monitoring each homodyne with a SR785 and maximizing or minimizing the observed spectrum.

Appendix A: Conventions

1. Squeezed States

A single-mode squeezed vacuum state, $|\xi\rangle$ can be generated from the vacuum, $|0\rangle$, by applying the squeezed operator, $\hat{S}(\xi)$, given by

$$\hat{S}(\xi) \equiv e^{\frac{1}{2}\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)}, \tag{A1}$$

such that $|\xi\rangle = \hat{S}(\xi)|0\rangle$. We find that the action of the squeezing operator on the raising and lowering operators is given by

$$\begin{aligned} \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) &= \cosh(r)\hat{a} - e^{i\theta_{\text{sqz}}}\sinh(r)\hat{a}^\dagger \\ \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) &= \cosh(r)\hat{a}^\dagger - e^{-i\theta_{\text{sqz}}}\sinh(r)\hat{a}, \end{aligned} \tag{A2}$$

where we have broken the complex constant ξ into a magnitude and phase-angle as $\xi = re^{i\theta_{\text{sqz}}}$.

With these conventions, the expected number of photons in a single-mode squeezed vacuum state, $\bar{n}$ is given by

$$\bar{n} \equiv \langle\xi|\hat{n}|\xi\rangle = \sinh^2(r), \tag{A3}$$

where the number operator is $\hat{n} = \hat{a}^\dagger\hat{a}$.

Equation (A2) usually contains all of the information we need to calculate properties of squeezed vacuum states in the Heisenberg frame. In the Schrodinger frame, we want to compute this state's first and second-order moments with respect to the position and momentum field quadratures given by

$$\begin{aligned} \hat{q} &\equiv \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \\ \hat{p} &\equiv \frac{i\left(\hat{a}^\dagger - \hat{a}\right)}{\sqrt{2}} \end{aligned} \tag{A4}$$

The first-order moments are given by

$$\begin{aligned}\langle\xi|\hat{q}|\xi\rangle &= 0\\ \langle\xi|\hat{p}|\xi\rangle &= 0,\end{aligned} \tag{A5}$$

and the (symmetrized) second-order moments are given by

$$\begin{aligned}\langle\xi|\hat{q}^2|\xi\rangle &= \frac{1}{2}\left[\cosh(2r) - \cos(\theta)\sinh(2r)\right]\\ \langle\xi|\hat{p}^2|\xi\rangle &= \frac{1}{2}\left[\cosh(2r) + \cos(\theta)\sinh(2r)\right]\\ \langle\xi|\{\hat{q},\hat{p}\}|\xi\rangle &= -\sinh(2r)\sin(\theta),\end{aligned} \tag{A6}$$

where $\{\hat{A},\hat{B}\} \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ denotes the anti-commutator of operators $\hat{A}$ and $\hat{B}$.

A squeezed state is a zero-mean Gaussian state and can be completely specified by its covariance matrix $\boldsymbol{V}_{\rm sqz}$ [2, 3]. The elements of this matrix can be read off from eq. (A6) and the matrix is given by

$$\boldsymbol{V}_{\rm sqz} = \frac{1}{2}\begin{bmatrix}\cosh(2r) - \cos\left(\theta_{\rm sqz}\right)\sinh(2r) & -\sinh(2r)\sin\left(\theta_{\rm sqz}\right)\\ -\sinh(2r)\sin\left(\theta_{\rm sqz}\right) & \cosh(2r) + \cos\left(\theta_{\rm sqz}\right)\sinh(2r)\end{bmatrix}. \tag{A7}$$

a. Decibels of Squeezing

Squeezed states are often quantified experimentally in terms of decibels of squeezing. Assuming a state has a squeezed position quadrature and an anti-squeezed momentum quadrature, its squeezing level in terms of decibels is given by

$$\begin{aligned}\mathrm{SQZ}_{\rm dB} &\equiv -10\log_{10}\left(\frac{\langle\xi|\hat{x}^2|\xi\rangle}{\langle 0|\hat{x}^2|0\rangle}\right)\\ &= 20\log_{10}(e)r\\ &\approx 8.686\times r.\end{aligned} \tag{A8}$$

The mean number of photons in a single squeezed mode is then given by

$$\bar{n} = \sinh^2\left(\frac{\mathrm{SQZ}_{\rm dB}}{8.686}\right). \tag{A9}$$

For instance, a 6 dB squeezed state has an expected photon number of about 0.55 and a 10 dB squeezed state has an expected photon number of about 2.

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